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Calculating Ratios of Harmonically Related, Complex Signals with Application to Nonlinear Large-Signal Scattering Parameters*

Jeffrey Jargon[1], Donald DeGroot[1], K.C. Gupta[2], and Alessandro Cidronali[3]
[1]National Institute of Standards and Technology
325 Broadway, M/S 813.01, Boulder, CO 80305, USA
Tel: 303.497.3596 | Fax: 303.497.3970 | E-mail: jargon@boulder.nist.gov
[2]University of Colorado at Boulder, Boulder, CO 80309, USA
[3]University of Florence, 50139 Florence, Italy

Abstract **- We describe a method for preserving time-invariant phase relationships when ratios are taken between any two harmonically related, complex signals. We provide a simple example to illustrate our technique, and show how this method is implemented when defining time-invariant nonlinear large-signal scattering parameters.**

I. Introduction

When two complex signals z and y exist at the same frequency $\omega/2\pi$, the ratio R of the two quantities may be expressed as

$$R = \frac{|z|}{|y|}\angle\left(\phi_z - \phi_y\right), \tag{1}$$

where the phasor notation of z is represented by $|z|\angle\varphi_z$ and that of y is represented by $|y|\angle\varphi_y$.

When two complex signals exist at different frequencies, obtaining a time-invariant phase of the ratio is more involved [1-2]. When ratios are taken between two harmonically related signals, we can preserve time-invariant phase relationships by introducing a third signal that acts as a phase reference. We show that this reference signal must have a component at the fundamental frequency in order that the ratios of any two harmonically related signals contain a time-invariant phase relationship. We provide a simple example to illustrate our technique. Finally, we show how this method is implemented when extracting nonlinear large-signal scattering parameters that are time-invariant.

II. Method

Consider two complex signals z_k and y_l that are harmonically related. Here k and l are positive integers representing signals at the kth and lth harmonic terms, respectively. In phasor form,

$$z_k = |z_k| \angle \phi_{z_k} \quad \text{and} \quad y_l = |y_l| \angle \phi_{y_l} \,. \tag{2}$$

Note that all phases φ considered here are in units of degrees (in terms of their respective frequencies) and have a modulus of 360° (i.e., $0° \leq \varphi < 360°$).

At first glance, a commonly assumed equation for taking the ratio of two harmonically related complex signals z_k and y_l is

$$R^{\mathrm{t}}_{kl} = \frac{|z_k|}{|y_l|} \angle \left(\phi_{z_k} - \frac{k}{l} \phi_{y_l} \right). \tag{3}$$

The factor k/l serves to translate the phase from the lth harmonic of the divisor to the kth harmonic of the dividend, resulting in the phase of the ratio R^{t}_{kl} given in terms of the kth harmonic. The superscript 't' is used because eq. (3) gives a time-variant phase. Specifically, if k/l is not an integer, there will be a phase ambiguity of $360°/l$.

In order to try to avoid a phase ambiguity, we modify eq. (3) by referencing the phases of signals z_k and y_l to some reference phase of a third signal x_n at the nth harmonic, which gives

$$R_{kl} = \frac{|z_k|}{|y_l|} \angle \left(\phi^{\mathrm{P}}_{z_k} - \frac{k}{l} \phi^{\mathrm{P}}_{y_l} \right), \tag{4}$$

where

$$\phi^{\mathrm{P}}_{z_k} = \phi_{z_k} - \frac{k}{n} \phi_{x_n} \quad \text{and} \quad \phi^{\mathrm{P}}_{y_l} = \phi_{y_l} - \frac{l}{n} \phi_{x_n} \,. \tag{5}$$

In eq. (5), the k/n factor serves to translate the arbitrary phase ($0° \leq \varphi_{x_n} < 360°$) from the nth harmonic of x_n to the kth harmonic of z_k and the l/n factor serves to translate the phase from the nth harmonic of x_n to the lth harmonic of y_l. Combining eqs. (4) and (5) gives

$$R_{kl} = \frac{|z_k|}{|y_l|} \angle \left[\left(\phi_{z_k} - \frac{k}{n} \phi_{x_n} \right) - \frac{k}{l} \left(\phi_{y_l} - \frac{l}{n} \phi_{x_n} \right) \right]. \tag{6}$$

Here, we still have the problem that if k/n or l/n are not integers, eq. (6) gives an inconsistent phase. Specifically, if k/n or l/n are not integers, there will be up to n ($n \leq l$) possible answers with a phase ambiguity of $360°/l$. If k is a multiple of l or vice versa, there will be fewer than n possibilities, but still more than one in general.

In order to avoid any phase ambiguity, k/n and l/n must be integers. In order for this to be true for all k and l, n must equal one. If the frequency of the reference signal (x_n) is set to its fundamental frequency ($n = 1$), then eq. (6) becomes

$$R_{kl} = \frac{|z_k|}{|y_l|} \angle \left[\left(\phi_{z_k} - k\phi_{x_1} \right) - \frac{k}{l} \left(\phi_{y_l} - l\phi_{x_1} \right) \right]. \tag{7}$$

Note that eq. (7) is algebraically identical to eq. (3). It is important, however, to leave the φ_{x_1} terms in and perform the phase references for φ_{z_k} and φ_{y_l} in order to ensure that eq. (7) provides a time-invariant phase. Eq. (7) can be simplified in the case of R_{k1} (l=1) if y_1 serves as both the divisor and the reference signal:

$$R_{k1} = \frac{|z_k|}{|y_1|} \angle \left(\phi_{z_k} - k\phi_{y_1} \right). \tag{8}$$

III. Example

Here, we provide a simple example illustrating that eq. (3) gives a time-variant phase, and that eq. (6) gives a time-variant phase for $n > 1$. However, eq. (6) does provide a time-invariant phase if $n = 1$.

In this example (see Figure 1), we consider signals with three phase references, the first one being arbitrary, where the reference at the fundamental is $x_1 = 1\angle 0°$, the

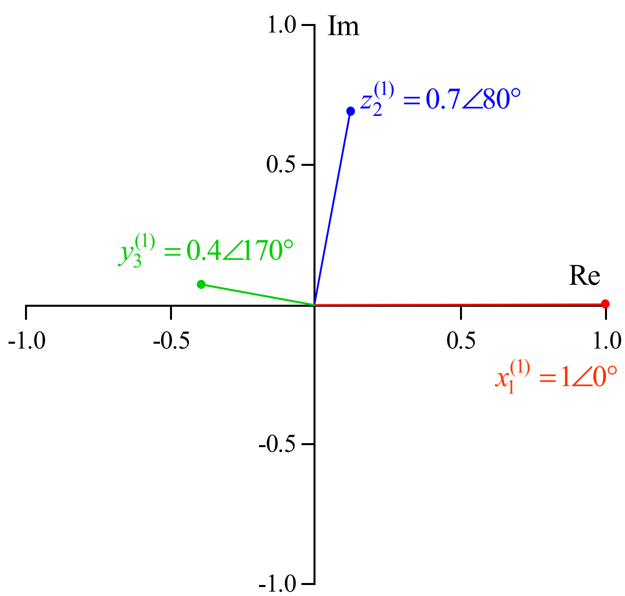

Figure 1. Phasor plot of the fundamental reference x_1, the dividend z_2, and the divisor y_3 at the first phase reference [all phasors identified by superscript (1)].

reference at the second harmonic is $x_2 = 1\angle 0°$, the dividend (at the second harmonic) is $z_2 = 0.7\angle 80°$, and the divisor (at the third harmonic) is $y_3 = 0.4\angle 170°$. Figure 2 shows the time-domain representation of x_1, z_2, and y_3. From the figure, we can see that the 80° phase delay in z_2 corresponds to a time delay of 0.111 fundamental-unit period, and the 170° phase delay in y_3 corresponds to a time delay of 0.157 fundamental-unit period. Figure 2 illustrates that there is no ambiguity in the time domain if all of the signals are synchronous with the fundamental signal. Equation (7) ensures the same in the frequency domain. But if, on the other hand, all of the signals are synchronous with a harmonic signal, portions of the waveforms at lower frequencies will be lost, resulting in possible phase ambiguities.

At the first phase reference, we determine R^{t}_{23} from eq. (3), which by definition uses y_3 as the reference in this case:

$$R^{t}_{23} = \frac{|z_2|}{|y_3|}\angle\left(\phi_{z_2} - \frac{2}{3}\phi_{y_3}\right). \tag{9}$$

Next, we calculate $R^{(2)}_{23}$ from eq. (6), using x_2 as the reference, as

$$R^{(2)}_{23} = \frac{|z_2|}{|y_3|}\angle\left[\left(\phi_{z_2} - \frac{2}{2}\phi_{x_2}\right) - \frac{2}{3}\left(\phi_{y_3} - \frac{3}{2}\phi_{x_2}\right)\right], \tag{10}$$

where the superscript '(2)' denotes the phase reference of x_2. Finally, we calculate $R^{(1)}_{23}$ from eq. (6), using x_1 as the reference, as

$$R^{(1)}_{23} = \frac{|z_2|}{|y_3|}\angle\left[\left(\phi_{z_2} - \frac{2}{1}\phi_{x_1}\right) - \frac{2}{3}\left(\phi_{y_3} - \frac{3}{1}\phi_{x_1}\right)\right], \tag{11}$$

where the superscript '(1)' denotes of phase reference of x_1. At this first phase reference, eqs. (9-11) give the same answer, of $1.75\angle 326.67°$, as shown in the first column of Table 1.

Next, we consider a second phase reference, where the phase of the fundamental frequency is shifted by 100°. This means that the phase at the second harmonic is shifted by 2 times 100°, or 200°, and the phase at the third harmonic is shifted by 3 times 100°, or 300°. So now, the reference at the fundamental is $x_1 = 1\angle 100°$, the reference at the second harmonic is $x_2 = 1\angle 200°$, the dividend is $z_2 = 0.7\angle 280°$, and the divisor is $y_3 = 0.4\angle 470° = 0.4\angle 110°$. These values are plotted in Figure 3. At this second phase reference, we again determine R^{t}_{23}, $R^{(1)}_{23}$, and $R^{(2)}_{23}$ using eqs. (9-11). Here, $R^{t}_{23} = 1.75\angle 206.67°$ is inconsistent with the answer determined at the first phase reference by 120° (360°/3). The ratio $R^{(2)}_{23} = 1.75\angle 326.67°$ is consistent with the answer determined

Table 1. Determining the ratios of z_2 to y_3 using three methods at three different phase references.

Quantity	1st Phase Reference	2nd Phase Reference	3rd Phase Reference
x_1	$1\angle 0°$	$1\angle 100°$	$1\angle 200°$
x_2	$1\angle 0°$	$1\angle 200°$	$1\angle 40°$
z_2	$0.7\angle 80°$	$0.7\angle 280°$	$0.7\angle 120°$
y_3	$0.4\angle 170°$	$0.4\angle 110°$	$0.4\angle 50°$
R^{t}_{23} [eq. (9)]	$1.75\angle 326.67°$	$1.75\angle 206.67°$	$1.75\angle 86.67°$
$R^{(2)}_{23}$ [eq. (10)]	$1.75\angle 326.67°$	$1.75\angle 326.67°$	$1.75\angle 206.67°$
$R^{(1)}_{23}$ [eq. (11)]	$1.75\angle 326.67°$	$1.75\angle 326.67°$	$1.75\angle 326.67°$

at the first phase reference. Likewise, the ratio $R^{(1)}_{23} = 1.75\angle 326.67°$ is also consistent with the answer determined at the first phase reference. The values of all of the quantities at the second phase reference are shown in the second column of Table 1.

Finally, we consider a third phase reference, where the phase of the fundamental frequency is shifted by 200°. This means that the phase at the second harmonic is shifted by 2 times 200°, or 400°, and the phase at the third harmonic is shifted by 3 times 200°, or 600°. So now, the reference at the fundamental is $x_1 = 1\angle 200°$, the reference at the second harmonic is $x_2 = 1\angle 40°$, the dividend is $z_2 = 0.7\angle 120°$, and the divisor is $y_3 = 0.4\angle 50°$. These values are plotted in Figure 4. At this third phase reference, we again determine R^{t}_{23}, $R^{(1)}_{23}$, and $R^{(2)}_{23}$ using eqs. (9-11). Here, $R^{t}_{23} = 1.75\angle 86.67°$ is inconsistent with the answers determined at the first and second phase references by 120° (360°/3). The ratio $R^{(2)}_{23} = 1.75\angle 206.67°$ is also inconsistent with the answers determined at the first and second phase references. The ratio $R^{(1)}_{23} = 1.75\angle 326.67°$, however, is consistent with the answers determined at the first and second phase references. The values of all of the quantities at the third phase reference are shown in the third column of Table 1.

Examining the fifth row of Table 1, we see that R^{t}_{23} does indeed give a time-variant phase. Since $k/l = 2/3$ is not an integer, there is a phase ambiguity of 360° / 3, or 120°. Examining the sixth row of Table 1, we see that that $R^{(2)}_{23}$ also gives a time-variant phase since the reference signal is located at the second harmonic. Since $l/n = 3/2$ is not an integer, there are 2 possible answers with a phase ambiguity of 360° / 3, or 120°. Examining the seventh row of Table 1, we see that that $R^{(1)}_{23}$ gives a time-invariant phase since the reference signal is located at the fundamental frequency. Thus, the only time-invariant ratio is $R^{(1)}_{23}$.

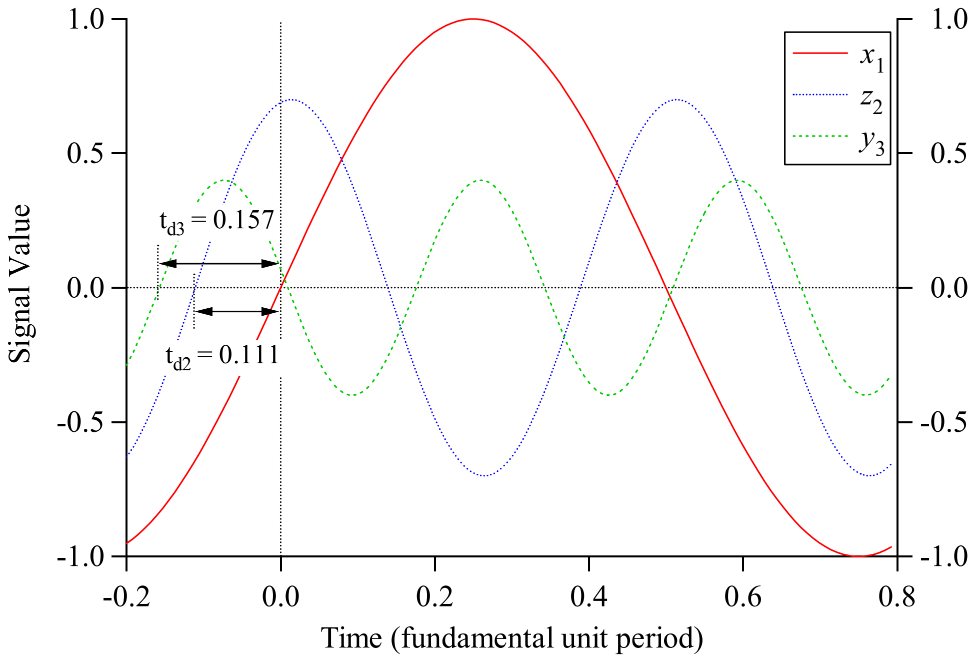

Figure 2. Time-domain plot of the fundamental reference x_1, the dividend z_2, and the divisor y_3 at the first phase reference.

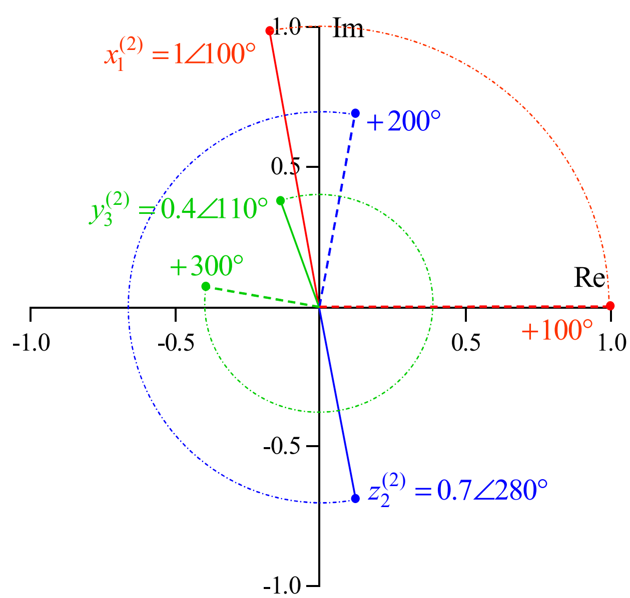

Figure 3. Phasor plot of the fundamental reference x_1, the dividend z_2, and the divisor y_3 at the second phase reference [using superscript (2)].

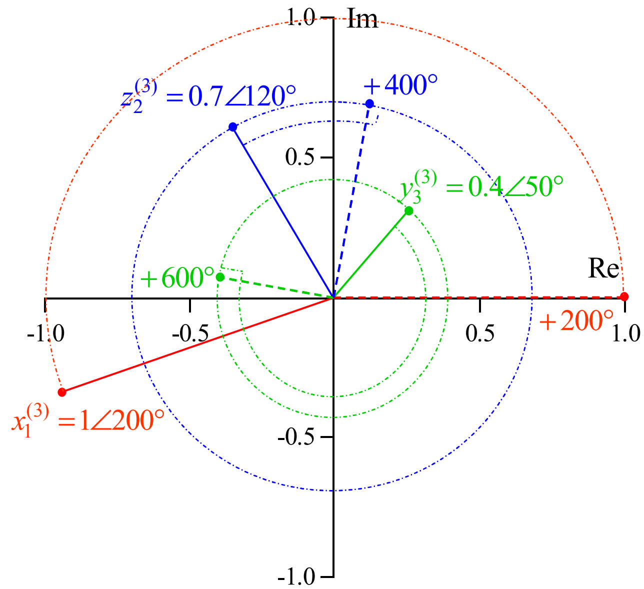

Figure 4. Phasor plot of the fundamental reference x_1, the dividend z_2, and the divisor y_3 at the third phase reference [using superscript (3)].

IV. Application to Nonlinear Large-Signal Scattering Parameters

In this section, we apply our methodology for preserving consistent phase relationships in ratios of two harmonically related signals to the definition of time-invariant nonlinear large-signal scattering parameters.

In previous work [3-4], we introduced the concept of nonlinear large-signal scattering parameters. Like commonly used linear *S*-parameters, nonlinear large-signal $\mathfrak{S}$-parameters can also be expressed as ratios of incident and reflected wave variables. However, unlike linear *S*-parameters, nonlinear large-signal $\mathfrak{S}$-parameters depend upon the signal magnitude and must take into account the harmonic content of the input and output signals since energy can be transferred to other frequencies in a nonlinear device.

For simplicity, we consider a two-port device excited at port 1 by a single-tone signal (a_{11}) at a frequency f_1. This condition is commonly encountered with power amplifiers and frequency doublers, although the approach can be generalized to any number of ports with multiple excitations that are harmonically related. In this case, we extract an input reflection coefficient

$$\mathfrak{S}_{11k1} = \frac{|b_{1k}|}{|a_{11}|}\angle\left(\phi_{b_{1k}} - k\phi_{a_{11}}\right)\Bigg|_{a_{mn}=0 \text{ for } \forall m \forall n\, [(m \neq 1)\wedge(n \neq 1)]}, \quad \textbf{(12)}$$

where a_{jl} (port j, spectral component number l) and b_{ik} (port i, spectral component number k) refer to the complex incident and scattered traveling voltage waves, respectively, and $\mathfrak{S}_{ijkl}$ indicates the nonlinear large-signal $\mathfrak{S}$-parameter. Instead of simply taking the ratio of b_{1k} to a_{11}, we phase reference to a_{11}. To do this we must subtract k times the phase of a_{11} from that of b_{1k}. This concept is identical to the simplified case presented in eq. (8), where a_{11} serves as both the reference and the divisor. The additional limitation imposed on eq. (12) is that all other incident waves other than a_{11} equal zero. Another valuable parameter, the forward transmission coefficient, is similarly extracted as

$$\mathfrak{S}_{21k1} = \frac{|b_{2k}|}{|a_{11}|}\angle\left(\phi_{b_{2k}} - k\phi_{a_{11}}\right)\Big|_{a_{mn}=0 \text{ for } \forall m \forall n\,[(m\neq 1)\wedge(n\neq 1)]}. \qquad \textbf{(13)}$$

This parameter provides the designer with a value of the gain or loss through a device, either at the fundamental frequency, or converted to a higher harmonic frequency.

Assuming a nonlinear model of a device or circuit exists, nonlinear large-signal $\mathfrak{S}$-parameters can be used to interrogate the model at certain conditions and provide a designer with useful engineering figures of merit. We provide an example where we look at the large-signal gain $\mathfrak{S}_{21k1}$ as a function of power for a nonlinear lumped-element model of a 2×90 μm GaAs pHEMT device operating at 5 GHz and a bias of V_{DS} = 3V and V_{GS} = -0.5 V [5-6], simulated using harmonic-balance with all a's other than a_{11} forced to zero. Figures 5 and 6 plot the magnitude and phase of $\mathfrak{S}_{2111}$ and $\mathfrak{S}_{2131}$ as a function of input power. We get a quantitative measure of the large-signal gain using nonlinear large-signal $\mathfrak{S}$-parameters. In Figure 5, we see that the magnitude of $\mathfrak{S}_{2111}$ rolls from 15.16 dB at $|a_{11}|$ = -20 dBm to 8.44 dB at $|a_{11}|$ = 10 dBm, while the phase of $\mathfrak{S}_{2111}$ remains nearly constant at around 135°. In Figure 6, we see that the magnitude of $\mathfrak{S}_{2131}$ increases from -46.55 dB at $|a_{11}|$ = -20 dBm to -5.95 dB at $|a_{11}|$ = 10 dBm, while the phase of $\mathfrak{S}_{2131}$ varies between -138° and -120°.

V. Concluding Remarks

We described a method for preserving time-invariant phase relationships when ratios are taken between two harmonically related signals by introducing a third signal that is used as a phase reference. We showed that a reference signal must be present at the fundamental frequency in order for time-invariant phase relationships to exist between ratios of any two harmonically related signals. We provided a simple example to illustrate our technique, and showed how this method is implemented when defining time-invariant nonlinear large-signal scattering parameters.

In the near future, we plan to examine whether this method can be generalized or modified to preserve consistent phase relationships when ratios are taken between two signals not harmonically related. In this case, a third signal occurs at a frequency that is a common factor of the first two and may not be readily available for use as a reference. Such a method could be very useful for mixer applications.

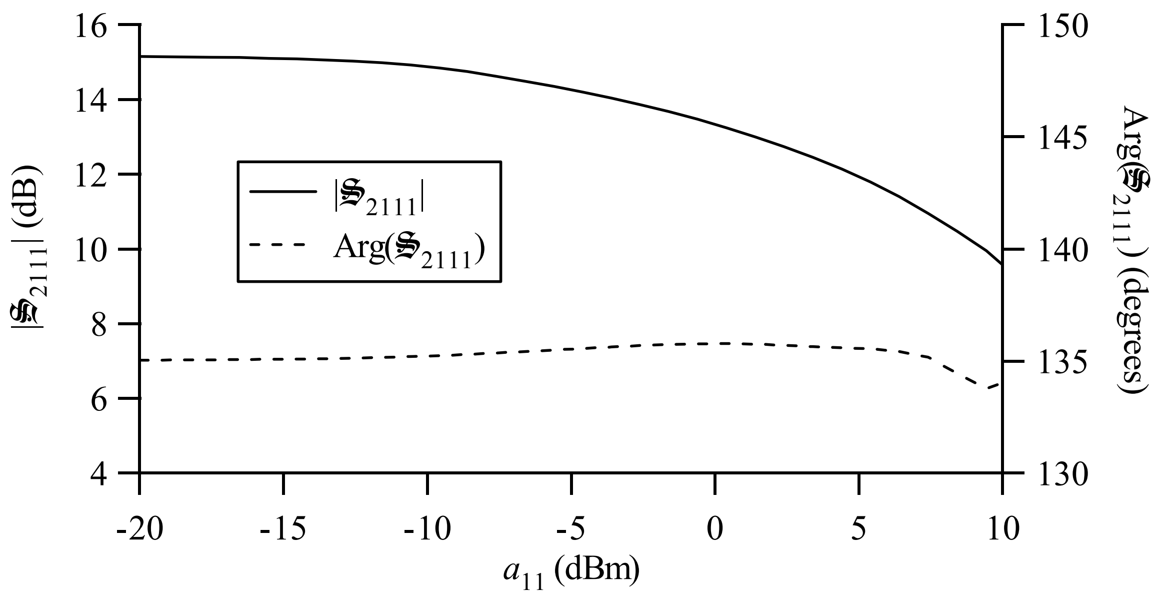

Figure 5. Magnitude and phase of 𝕾$_{2111}$ as a function of power for a nonlinear lumped-element model of a 2×90 µm GaAs pHEMT device operating at 5 GHz and a bias of V_{DS} = 3V and V_{GS} = -0.5 V.

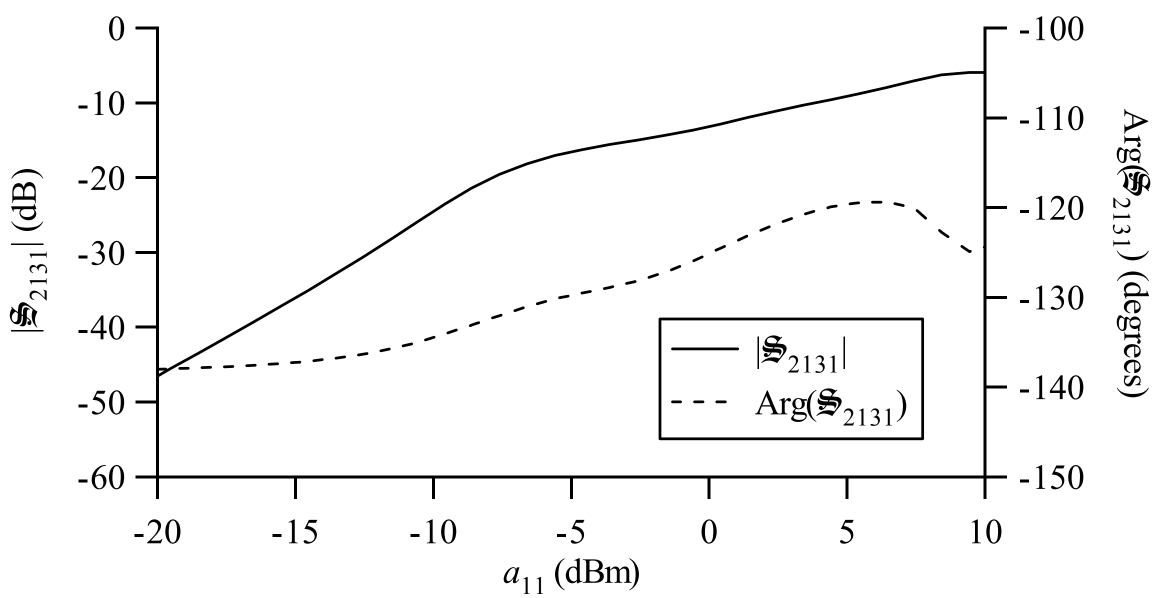

Figure 6. Magnitude and phase of 𝕾$_{2131}$ as a function of power for a nonlinear lumped-element model of a 2×90 µm GaAs pHEMT device operating at 5 GHz and a bias of V_{DS} = 3V and V_{GS} = -0.5 V.

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